Deviation of Light near the Sun in General Relativity

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http://www.lacosmo.com/DeflectionOfLight/DeflectionOfLight.html

The deviation of light rays near the Sun is one of the most dramatic predictions of general relativity and the verification of this effect by Eddington in 1919 brought a spectacular confirmation of Einstein's theory. But it is quite difficult to provide the reader with the details of the calculation as there is no simple way to do it. In fact, to get the result it is necessary to dive into the full theory of general relativity. Here is a good opportunity to discover this theory while illustrating it on an example!

The derivation of the equations given here closely follows the presentation of Edwin F. Taylor and John Archibald Wheeler in their delightful work "Exploring Black Holes, Introduction to General Relativity" (Addison Wesley Longman, 2000).

1 Principle of the calculation

Whereas the qualitative aspect and the historical impact of the phenomenon of light bending near the Sun are described in many places, the calculation of the angle of deviation is rarely given. This page is dedicated to this numerical exercise.

General relativity teaches us that by propagating into space a photon follows what is called a "geodesic" of spacetime. Thus we have to examine the following points :

- what is a "geodesic" ?
- which equations govern a geodesic ?
- find the solutions of those equations and thus discover the trajectory of light through space ;
- compute the angle between the initial and final directions of light.

2 Metrics for curved spacetime around a massive object

Newtonian mechanics describes the motion of a particle in an absolute space with respect to an absolute time. The position of the moving object is located by its coordinates with respect to some frame of reference and is given as a function of time t. General relativity affirms that there exists no absolute time ant that time cannot be dissociated from space. The theory bases its reasoning on *events*, each event being characterized by a point M (where it happens !) and a time t (when it happens !). The events attached to a moving particle constitute what is called a *worldline*.

Let us consider for example a spaceship moving freely through space, which means that all his motors are turned off. Let us imagine that regular flashes are emitted in accordance with a clock located inside the rocket and beating the time. The time interval between two successive flashes will be denoted by τ (this quantity is thus measured with respect to the proper time of the spaceship). Think now of another frame of reference as constituted by an ensemble of space beacons, also free from acceleration, at constant mutual distances from one another (each free beacon stays at the same distance from its neighbours). Every signal bears the indication of its position in space (for instance by showing its distance from some origin) and holds its own clock. The clocks of this second frame are synchronized between them. Then in that frame the interval between two flashes (i.e. two events) is characterized by two numbers : the space interval s and the time interval t. To determine those two quantities it suffices to record which beacon faces Flash #1 and which beacon faces Flash #2 while noting the times of those events.

Special relativity is based on the following principle. The proper time interval τ between Event #1 et Event #2 is given by the formula

$$\tau^2 = t^2 - s^2 \tag{1}$$

and this quantity does not depend on the frame in which it is evaluated. In other words all observers agree on the value of τ computed by Formula (1), although the values of s and t differ from one system of reference to another.

Be careful : unless otherwise indicated, distances will be measured in units of time, as is often done in astronomy. We have chosen to do so in writing Equation (1). On the contrary if distances s are expressed in conventional units, for instance in centimeters, then one should pass from the latter to our distance s expressed in seconds via the formula $s(\text{in seconds}) = s(\text{in centimeters})/c$ where c is the speed of light in conventional units, namely 3×10^{10} cm/s. (Expressing distances and times with the same unit would amount to taking the speed of light equal to unity.)

In general relativity the property of invariance of the proper interval with respect to a change of the coordinates remains valid but only locally, i.e. under the condition of staying in a sufficiently small region of spacetime (its size depends on the accuracy of the measurements). The main novelty concerns the expression of the proper time as given by Formula (1). The coefficients entering this formula depend now on the point of spacetime under consideration and the resulting expression takes the name of *metrics*. In fact the whole structure of spacetime, and especially its curvature, is included in the local expression of τ and in the form of its coefficients.

We are interested here in the structure of spacetime around the sun. In order to describe the physics locally we consider two nearby events separated by infinitesimal amounts of the time and space coordinates dt, dx, dy and dz. If space were flat, the metrics would have the form

$$(d\tau)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

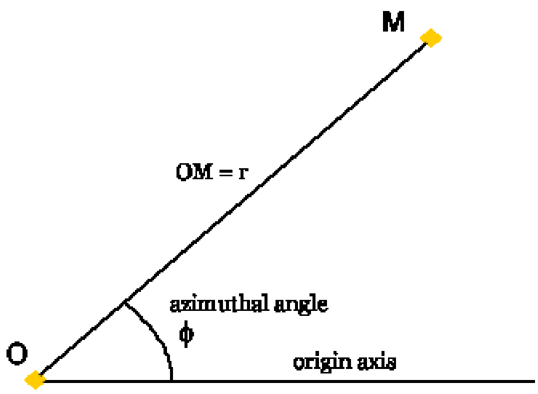
which is usually written (and a little sloppily) by convention as

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{2}$$

By working in spherical coordinates, in a plane containing the center of the sun (this choice removes one spatial coordinate), that formula becomes

$$d\tau^2 = dt^2 - dr^2 - r^2 d\phi^2$$

where r denotes the distance to the center and ϕ an azimutal angle in the plane of the orbit (see the figure below).



Polar coordinates (r, ϕ) of point M
with respect to origin O

But spacetime around a center of attraction of mass M (for instance a black hole or the vicinity of the sun) is not flat. It is characterized by the *Schwarzschild metric*

$$d\tau^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2 d\phi^2 \tag{3}$$

The story is really fantastic : the *whole* structure of spacetime is embodied in this "simple" formula (3).Even the famous black hole lurks behind those apparently innocuous symbols.

One question : in which units is expressed the mass M in that formula ? It is seen that M has the dimension of a length, a quantity that we measure here in seconds. Therefore M will also be measured in seconds. The formula allowing to transform grams in seconds is

$$M(\text{in seconds}) = (G/c^3)M(\text{in grams})$$

where $(G/c^3) = 2,5 \times 10^{-39}$ s/g.

3 The equations of a geodesic

The metric, that is (exactly) the formula expressing at a givent point of spacetime the temporal interval between two nearby events, reveals the presence of curvature as soon as the expression deviates from Formula (2) corresponding to flat euclidian space. That metric will allow us to find the properties of the motion of a test particle free from acceleration. Actually both special and general relativity teach us that between two given events "E_1 and E_2 a freely moving body follows the path for which the time interval τ is maximum. Equivalently one can say that a freely moving particle follows a geodesic of spacetime as a geodesic is precisely defined by this property of maximazing the time interval.

> **Definition of a geodesic :** *the geodesic between two events E_1 and E_2 is the wordline for which the interval of proper time between E_1 and E_2 is maximum.*

That property of maximazing the proper time will allow us to derive the equations of a geodesic. It will also yield the expressions of the energy and angular momentum of a particle in orbit around the center of attraction.

4 Energy of the particle

Let us apply the principle of maximisation of the proper time interval in the following manner. Suppose that a free spatial ship (whose rockets are turned off) falls radially, therefore along a straight path, towards the central attractive mass. Imagine that three successive flashes, with nearby time and space coordinates, are emitted inside the spaceship. We observe those three events in some external frame. In that latter frame the event E_1 consists in the emission of a flash at time $t = 0$ when the spatial engine is located at radius r_1. The flash E_2 is emitted at time t when the cabin is at radius r_2. The flash E_3 is emitted at time T when the cabin is at radius r_3. The quantity T is assumed to be small. We then assume that we vary the intermediate coordinates of E_2. The principle of maximal aging says that the geodesic starting from E_1 and ending at E_3 will pass through Event E_2 such that the proper time interval

$$\tau = \tau_A + \tau_B, \tag{4}$$

is maximum. Here τ_A measures the interval over the first spacetime segment A, which connects E_1 to E_2 and τ_B measures the time interval over the second segment B, which connects E_2 to E_3.

In order to avoid varying all quantities at the same time, we assume in this experiment that the locations of the radii r_2 and r_3 are fixed and that only the time t, at which the second flash is emitted, is allowed to change. According to Formula (3) the interval of proper time over the first segment A is given by its square

$${\tau_A}^2 = (1 - 2M/r_A)t^2 + (\text{terms without } t) \tag{5}$$

from which we deduce

$$\tau_A d\tau_A = (1 - 2M/r_A)tdt \tag{6}$$

The lapse of time over Segment B between the events E_2 and E_3 is $(T - t)$, and therefore the proper time duration τ_B is given by

$${\tau_B}^2 = (1 - 2M/r)(T - t)^2 + (\text{terms without } t) \tag{7}$$

from which we deduce

$$\tau_B d\tau_B = -(1 - 2M/r_B)(T - t)dt. \tag{8}$$

To make the total time interval $\tau = \tau_A + \tau_B$ maximum with respect to a variation dt of the time t, we write

$$\frac{d\tau}{dt} = \frac{d\tau_A}{dt} + \frac{d\tau_B}{dt} = 0 \tag{9}$$

Deducing $d\tau_A$ and $d\tau_B$ from Equations (6) and (8) and letting quite naturally $t = t_A$ and $T - t = t_B$, we easily get

$$(1 - 2M/r_A)(t_A/\tau_A) = (1 - 2M/r_B)(t_B/\tau_B)\ . \tag{10}$$

The left side of that equation depends only on parameters characterizing the first segment A (which connects E_1 to E_2). The right side depends only on parameters related to the second segment B (which connects E_2 to E_3).

We have discovered in Equation (10) a quantity that is the same for both segment. This quantity is thus a constant of the motion for the free particle under consideration. For good physical reasons (especially to recover the formulae of special relativity), one is led to identify that constant of motion as the ratio of the energy of the particle to its mass. We write this very important result under the form

$$E/m = (1 - 2M/r)(dt/d\tau) \tag{11}$$

an expression in which we have returned to the differential notation for the intervals t and τ.

Incidentally we may notice that with the units we have chosen, energy E and mass M are expressed in the same unit (for instance the centimeter).

5 Angular momentum of the particle

We have applied the principle of maximazing the proper time interval by varying the time of the intermediate event E_2. We now perform the same operation but this time we vary the angle ϕ of that intermediate event. We recall that ϕ measures the direction of the moving particle with respect to some direction chosen as the origin. We call it the *azimuth*.

We consider again three events consisting in the emission of flashes inside a spaceship floating freely in space. The first segment A connects Event E_1 to Event E_2. The second segment B connects E_2 to E_3. The azimuthal angle of the first event is fixed at $\phi = 0$. The angle of the last one is fixed at $\phi = \Phi$. The intermediate azimuth is taken as the variable ϕ. Again in order not to vary everything at the same time, we assume that the radius r at which the second flash is emitted stays constant.

We follow the same chain of reasoning as in the previous section. From the metric (3), the time interval τ_A over the first segment is given by its square

$$\tau_A{}^2 = -r_A{}^2\phi^2 + (\text{terms without } \phi) \tag{12}$$

and the interval τ_B over the second by

$$\tau_B{}^2 = -r_B{}^2(\Phi - \phi)^2 + (\text{terms without } \phi) \tag{13}$$

from which we get

$$\begin{aligned} \tau_A d\tau_A &= -r_A{}^2\phi\, d\phi && (14)\\ \tau_B d\tau_B &= r_B{}^2(\Phi - \phi)d\phi && (15) \end{aligned}$$

By writing $d\tau/d\phi = d(\tau_A + \tau_B)/d\phi = 0$ one easily obtains, similarly to Formula (10)

$$r_A{}^2\phi_A/\tau_A = r_B{}^2\phi_B/\tau_B \tag{16}$$

after having written quite naturally $\phi = \phi_A$ and $\Phi - \phi = \phi_B$. The left side, which contains only terms that are specific to the first segment, is equal to the right side, which contains only terms relative to the second segment. We thus exhibit another constant of motion, namely $r^2 d\phi/d\tau$ (by shifting back to the differential notation), a quantity that turns out to be identified with the ratio of the angular momentum L of the particle to its mass m, which we write as

$$L/m = r^2(d\phi/d\tau) \tag{17}$$

6 Computing the orbit

Technically speaking in order to determine the trajectory of a moving body free from acceleration we apply the following strategy. Knowing the energy E and the angular momentum L of the particle of mass m (E and L depend on the initial conditions) we can follow the position of that particle by computing the increments of its spacetime coordinates t, r and ϕ as the proper time τ itself advances. Algebraically for each increment $d\tau$ of the proper time we compute (or the computer calculates) the corresponding increments dt, dr and $d\phi$ of the coordinate of the mobile body. The squares of the increments dt and $d\phi$ are extracted from Equations (11) and (17) in the following form :

$$\begin{aligned} dt^2 &= (E/m)^2(1-2M/r)^{-2}d\tau^2 \qquad (18)\\ d\phi^2 &= (L/m)^2 r^{-4} d\tau^2 \qquad (19)\end{aligned}$$

We notice that the expression of dr is missing. We get it by transporting the values of dt and $d\phi$ into the metric equation (3) and solving it for dr. This yields

$$dr^2 = \left\{(E/m)^2 - (1-2M/r)[1+(L/m)^2 r^{-2}]\right\} d\tau^2 \qquad (20)$$

By dividing both sides of Equations (20) and (19) we directly arrive to the equation of the orbit in polar coordinates as

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \left(\frac{E}{L}\right)^2 - \left(1-\frac{2M}{r}\right)\left[\left(\frac{m}{L}\right)^2 + \frac{1}{r^2}\right] \qquad (21)$$

7 The trajectory of light

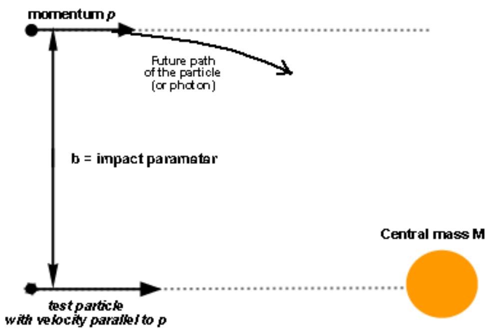
The preceding treatment is apparently not relevant to the case of a photon. In fact the calculation of the trajectory was done by incrementing the proper time but this latter concept has no meaning for a photon since the interval between two events that are located on the wordline of a photon is always equal to zero (as at light velocity $s = t$, the interval $\tau^2 = t^2 - s^2$ vanishes).

Nevertheless it happens that by letting the mass of the particle tend towards zero, one arrives at the right results. Thus for $m = 0$ our equation (21) takes the form

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \left(\frac{E}{L}\right)^2 - \left(1-\frac{2M}{r}\right)\frac{1}{r^2} \qquad (22)$$

That equation will allow us to determine the deviation of ligth rays passing near the sun.

It is necessary to specify the parameters found in the formulae. First the angular momentum of the moving particle at infinity is equal by definition to the product of its linear momentum p by what is called the *impact parameter* b, which represents the distance between the center of attraction (the sun in the present case) and the initial direction of the velocity of the particle (see the figure).



In other words

$$L = p\, b \tag{23}$$

In addition it is known that the momentum p of a photon is equal to its energy E (with the units that were chosen). It results at once from this formula that

$$L/E = b\ . \tag{24}$$

If the ratio L/E is equal to the impact parameter b Equation (22) writes as

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{b^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2} \tag{25}$$

8 Determination of the deflection angle

Formula (25) will allow us to determine the change in the direction of a light pulse caused by the gravitational field of the sun. To achieve this aim we have to sum up the successive infinitesimal increments $d\phi$ of the azimuthal angle ϕ along the path. This means that we have to carry out the integration of $(d\phi/dr)dr$ when r varies from the minimum distance denoted R (R is the radius of the sun if the light ray grazes its surface). We should still multiply that quantity par 2 to account for both symmetrical "legs" of the trajectory (the photon first approaches the Sun then recedes from it).

It is necessary to stipulate a further point, namely the relation existing between the two quantities b and R that we have introduced, and that are not independent. The point $r = R$ corresponds to the place where the light photon is closest to the sun. There the photon moves tangentially. Since at that point there is no radial component, we can write that the derivative dr/dt vanishes. It suffices to take the element dr from Equation (25) to find immediately

$$\frac{1}{b^2} = \left(1 - \frac{2M}{R}\right)\frac{1}{R^2} \tag{26}$$

so that this same equation (25) becomes

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \left(1 - \frac{2M}{R}\right)\frac{1}{R^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2} \tag{27}$$

The form of the expression dictates to us to pose

$$u = R/r$$

where u varies between 1 and 0. The last equation (27) then becomes

$$\begin{aligned} (du/d\phi)^2 &= (1 - 2M/R) - (1 - 2Mu/R)u^2 \\ &\text{or} \\ (du/d\phi)^2 &= 1 - u^2 - (2M/R)(1 - u^3) \end{aligned} \tag{28}$$

Consequently the infinitesimal variation $d\phi$ of the azimuth is given in terms of the variation du of R/r by

$$\begin{aligned} d\phi &= [1 - u^2 - (2M/R)(1 - u^3)]^{-1/2}\, du \\ &= \frac{(1 - u^2)^{-1/2} du}{[1 - (2M/R)(1 - u^3)(1 - u^2)^{-1}]^{1/2}} \end{aligned} \tag{29}$$

The presence of the term $(1 - u^2)$ in Expression (29) encourages us to make the change of variable

$$u = \cos\alpha,\ 0 < u < 1,\ 0 < \alpha < \pi/2$$

which leads to

$$d\phi = \left[1 - (2M/R)(1 - \cos^3\alpha)\sin^{-2}\alpha\right]^{-1/2} d\alpha \tag{30}$$

By observing that

$$\frac{1 - \cos^3\alpha}{\sin^2\alpha} = \frac{(1 - \cos\alpha)(1 + \cos\alpha + \cos^2\alpha)}{(1 - \cos\alpha)(1 + \cos\alpha)} = \cos\alpha + \frac{1}{1 + \cos\alpha}$$

we end up with the final equation of the trajectory under the form

$$d\phi = \left[1 - (2M/R)\left(\cos\alpha + \frac{1}{1 + \cos\alpha}\right)\right]^{-1/2} d\alpha \tag{31}$$

avec

$$\cos\alpha = R/r$$

It is interesting to emphasize that so far there have been no approximation. This is quite rewarding.

9 Approximations and integration

The small value of the term M/R will allow us to make an approximation and in this way will make us able to complete the integration. In conventional units the mass of the sun is 2×10^{33} grams and its radius is 7×10^{10} centimeters. By using the factor $G/c^2 = 7,4 \times 10^{-29}$ cm/g which makes it possible to transform grams into centimeters, we get

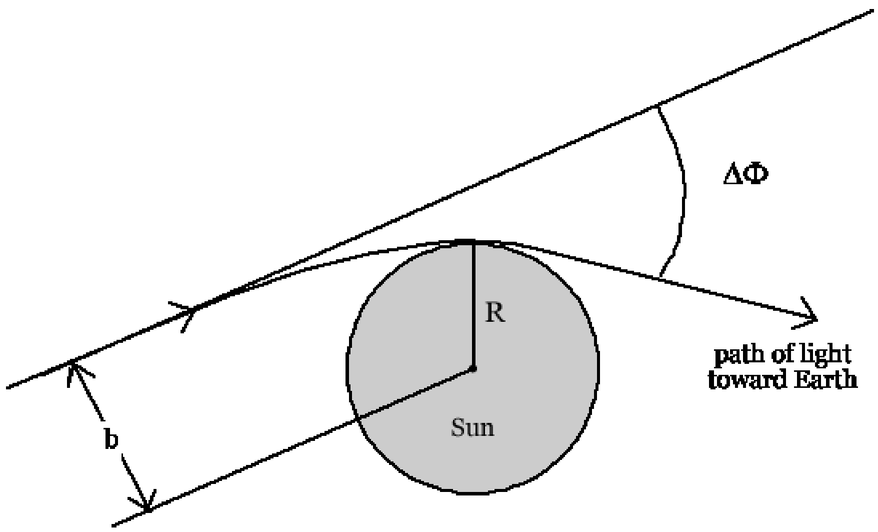
$$M/R = (G/c^2)M(\text{in grams})/R(\text{in centimeters}) = 2 \times 10^{-6}$$

In Equation (31) we can thus use the classical approximation $(1+\epsilon)^p \simeq 1+p\,\epsilon$ to arrive at

$$d\phi = \left[1 + (M/R)\left(\cos\alpha + \frac{1}{1+\cos\alpha}\right)\right] d\alpha \tag{32}$$

Therefore the total variation of the azimuth ϕ along the path of the photon is

$$\begin{aligned} \phi &= 2\int_0^{\pi/2} \left[1 + (M/R)\left(\cos\alpha + \frac{1}{1+\cos\alpha}\right)\right] d\alpha && (33)\\ &= 2\left[\alpha + \frac{M}{R}\left(\sin\alpha + \tan\frac{\alpha}{2}\right)\right]_0^{\pi/2} && (34)\\ &= \pi + 4M/R && (35) \end{aligned}$$



gravitational deflection angle ΔΦ of starlight by Sun
(greatly exaggerated)

The first term π gives the total change in the azimuthal angle of the photon where there is no Sun present, since in that case the photon follows a straight path. It is the second term that gives the additional angle of deflection $\Delta\phi$ with respect to this straight line

$$\Delta\phi = 4M/R \tag{36}$$

or in conventional units

$$\Delta\phi = 4(G/c^2)M(\text{grams})/R(\text{centimeters}) \tag{37}$$

Numerically at the surface of the sun (with the values of the mass and the radius given above) one finds $\Delta\phi = 8,5 \times 10^{-6}$ radian, or (knowing that π radians equal 180 degrees and that there are 60 minutes of arc in one degree and 60 seconds of arc in one minute of arc)

$$\Delta\phi = 1,75"$$